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Enumeration by kernel positions ☆

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ABSTRACT

We introduce a class of two-player games on posets with a rank function, in which each move of the winning strategy is unique. This allows one to enumerate the kernel positions by rank. The main example is a simple game on words in which the number of kernel positions of rank n is a signed factorial multiple of the nth Bernoulli number of the second kind. Generalizations to the degenerate Bernoulli numbers and to negative integer substitutions into the Bernoulli polynomials are developed. Using an appropriate scoring system for each function with an appropriate Newton expansion we construct a game in which the expected gain of a player equals the definite integral of the function on the interval [0, 1].

0. Introduction

Enumerative combinatorics is often used to prove that a sequence of numbers consists of positive integers, by showing that the nth entry in the sequence counts some patterns of order n. Recursive formulas for positive integers may be shown by exhibiting a recursive structure of the counted objects. It is not unusual either to see a sequence of integers with alternating signs, and as-

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sociate some inclusion–exclusion formula to them. From this perspective the Bernoulli numbers b_n of the second kind represent a remarkable puzzle. These are rational numbers, but $(n+1)!b_n$ is an integer for all n. The numbers b_0 and b_1 are positive, for higher values of n the sign of b_n alternates. They satisfy a recursion formula that does not resemble neither inclusion–exclusion, nor formulas associated to objects with a recursive structure. Intuitively, the formula suggests introducing "Bernoulli objects of rank n" as a set of pairs of permutations which "contain no Bernoulli object of any lower rank." Expanding such a recursive definition leads to a formula with alternating existential and universal quantifiers, naturally associated with the notion of a *two-player game*.

The core idea of this paper may be found in Section 2 where we introduce a very simple game on pairs of words in such a way that the probability that the second player has a winning strategy starting from a random position of rank n is $(-1)^{n-1}b_n/(n+1)!$. Such positions are called *kernel positions* in the Sprague–Grundy theory of two-player games. We are able to calculate this probability by two reasons. The first reason is that, by the nature of the game, from each non-kernel position there is exactly one kernel position that is reachable. The second reason is that each position of rank $m<n$ is reachable from the same number $\gamma_{m,n}$ of positions of rank n in a single step. Therefore the total number of positions of rank n may be expressed as a sum over m where m is the rank of the unique kernel position reachable from our starting non-kernel position of rank n in a single step. This idea of *"enumeration by kernel positions"* is explored at a reasonably high level of generality in Section 3. Most applications presented in this paper involve games on (pairs, triplets of) words where a valid move involves taking some initial segment, but the framework we develop is applicable to a wider range of combinatorial situations. We also explicitly describe all possible games that are played once a starting position $x \in P$ is selected, together with an explicit description of the winning strategy.

In Section 4 we introduce a variant of the original Bernoulli game whose kernel positions are enumerated by negative integer substitutions into the Bernoulli polynomials of the second kind. We also define a game that turns out to be associated to the Bernoulli polynomials of the first kind, although this connection is more remote. This is not surprising, considering the fact that the Bernoulli numbers of the first kind have a very different signature pattern from the Bernoulli numbers of the second kind.

There is a plethora of formulas in the mathematical literature evaluating finite and infinite linear combinations of Bernoulli polynomials and numbers of the second kind. In Section 5 we demonstrate how we may use these formulas to predict the expected gain of a player when two players play several rounds of the original or polynomial Bernoulli games (of the second kind) and pay money according to various scoring systems. In particular, we show how to construct a scoring system in which the expected gain is the definite integral of a given function on the interval $[0,1]$, and another one where the expected gain equals Euler's constant C. Many of our examples follow a signature pattern that is ideally suited to play the game in a casino: at the beginning, the gambler is supposed to pay a specific amount to the casino for the right to play a potentially infinite sequence of Bernoulli games of ever increasing difficulty, for ever decreasing payouts.

The Bernoulli numbers and polynomials of the second kind were generalized by Carlitz [4], and the study of these *degenerate* Bernoulli polynomials and numbers is still ongoing. In Section 6 we construct a refinement of the original Bernoulli game, in which the enumeration of kernel positions is related to these degenerate Bernoulli numbers. Generalizing this game to a game that is related to the degenerate Bernoulli polynomials seems to be a hard question, since the degenerate Bernoulli polynomials also generalize the Bernoulli polynomials of the first kind, for which the best model in the framework of our theory is yet to be found.

The idea of enumeration by kernel positions is more general than creating models for Bernoulli numbers and polynomials and their generalizations. This fact is illustrated by the analysis of two very simple games in Section 7, where the simplest game in this paper turns out to have a kernel-position generating function that is most easily expressed in terms of the highly non-trivial digamma function. Further generalizations and questions are outlined in the concluding Section 8.

1. Preliminaries

1.1. Bernoulli polynomials, numbers, and some generalizations

The usual definition of the *Bernoulli polynomials $B_n(x)$ of the first kind* is equivalent to stating

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{t \cdot e^{xt}}{e^t - 1}, \tag{1}$$

a sample reference is Roman [12, p. 30]. One of our major references, Jordan's book [7], uses a slightly different terminology. Jordan's [7, p. 250] Bernoulli polynomials $\phi_n(x)$ of the first kind satisfy

$$\sum_{n=0}^{\infty} \phi_n(x) t^n = \frac{t \cdot e^{xt}}{e^t - 1}.$$

We have thus $\phi_n(x) = B_n(x)/n!$. The *Bernoulli numbers B_n of the first kind* are the Bernoulli polynomials of the first kind evaluated at zero: $B_n = B_n(0)$. (Jordan's $B_n = n! \cdot \phi_n(0)$ gives the *same numbers*, whereas his definition for the nth Bernoulli number of the second kind is still off by a factor of $n!$, see below.) The Bernoulli polynomials may also be defined by the property

$$\sum_{k=0}^{x-1} k^n = \frac{1}{n+1} \big(B_{n+1}(x) - B_{n+1}(0)\big). \tag{2}$$

This was the property used by Bernoulli [3].

The usual definition of the *Bernoulli polynomials $b_n(x)$ of the second kind* is equivalent to stating

$$\sum_{n=0}^{\infty} \frac{b_n(x)}{n!} t^n = \frac{t(1+t)^x}{\ln(1+t)}, \tag{3}$$

see Roman [12, p. 116]. Jordan's Bernoulli polynomials $\psi_n(x)$ of the second kind [7, p. 279] satisfy

$$\sum_{n=0}^{\infty} \psi_n(x) t^n = \frac{t(1+t)^x}{\ln(1+t)},$$

thus we have $\psi_n(x) = b_n(x)/n!$. Roman makes an explicit note of this difference [12, p. 114].

One reason to like Jordan's variant is that the polynomials $\phi_n(x)$ and $\psi_n(x)$ exhibit a duality between the use of the operators d/dx and Δ. Here Δ is the *finite difference operator*, given by $\Delta p(x) = p(x+1) - p(x)$. In fact, Jordan [7, §78] defines the polynomials $\phi_n(x)$ as the sums

$$\phi_n(x) = \Delta^{-1} \frac{x^{n-1}}{(n-1)!}$$

subject to the initial condition

$$\frac{d}{dx} \phi_n(x) = \phi_{n-1}(x).$$

Sequences of polynomials $\{F_n(x)\}$ satisfying $\frac{d}{dx} F_n(x) = F_{n-1}(x)$ may be written in the form

$$F_n(x) = a_0 \frac{x^n}{n!} + a_1 \frac{x^{n-1}}{(n-1)!} + \cdots + a_n$$

where the coefficients $a_0, a_1, \ldots$ are the same for all n. For the coefficients a_n associated to the polynomials $\phi_n(x)$, Jordan defines $B_n := a_n \cdot n! = n! \cdot \phi_n(0)$ and obtains the usual Bernoulli numbers of the first kind.

Dually, Jordan defines his Bernoulli polynomials $\psi_n(x)$ of the second kind [7, §89] as the antiderivatives

$$\psi_n(x) = \int \binom{x}{n-1}\,dx$$

subject to the initial condition that is implied by

$$\Delta\psi_n(x) = \psi_{n-1}(x).$$

As explained in [7, §22], all elements in a sequence of polynomials $\{F_n(x)\}$ satisfying $\Delta F_n(x) = F_{n-1}(x)$ may be given by a general formula

$$F_n(x) = c_0\binom{x}{n} + c_1\binom{x}{n-1} + \cdots + c_n\binom{x}{0}$$

where the coefficients $c_0, c_1, \ldots$ are the same for all n. For the polynomials $\psi_n(x)$ Jordan calls these coefficients c_n Bernoulli numbers of the second kind, and uses the notation b_n. We will *not* follow him at this point, since in general we have $c_n = F_n(0)$, thus we may write

$$\psi_n(x) = \psi_0(0)\binom{x}{n} + \psi_1(0)\binom{x}{n-1} + \cdots + \psi_n(0)\binom{x}{0}. \tag{4}$$

Therefore Jordan's Bernoulli numbers $\psi_n(0)$ of the second kind are related to the usual definition by $\psi_n(0) = b_n/n!$.

Jordan proves that the numbers $\{\psi_n(0)\}$ satisfy the equation

$$\sum_{m=0}^{n-1}(-1)^m\frac{\psi_m(0)}{n-m} = 0. \tag{5}$$

This is Eq. (4) in [7, p. 266]. (Letting the summation go to n would contribute a meaningless term, but Jordan did not mean the last term to be read, see our next remark.)

Remark 1.1. At a first glance, in Jordan's summation formulas there appears to be an extra summand contributing zero or even a meaningless term. This is due to the fact that Jordan summation notation is different from the ordinary notation. This is first pointed out in a footnote on p. 8 of [7], where Jordan warns that the upper limit of his sums should never be included in the evaluation. The explanation for this choice of notation may be found in [7, §40]. We have no reason to abide by Jordan's unorthodox summation notation, but we have to be aware of it when we cite his summation formulas in a modern publication.

Using (5) as a recursion formula for $\psi_n(0)$ and the initial condition $\psi_0(0) = 1$, the numbers $\psi_n(0)$ are completely determined.

Bernoulli polynomials and numbers of both kinds have been generalized in many different ways. One important generalization is the introduction of *degenerate Bernoulli polynomials* $\beta_m(\lambda, x)$ by Carlitz [4] (see also [5]). These are given for $\lambda \neq 0$ by

$$\sum_{n=0}^{\infty}\beta_n(\lambda, x)\frac{t^n}{n!} := \left(\frac{t}{(1+\lambda t)^{\mu} - 1}\right)(1+\lambda t)^{\mu x} \quad \text{where } \lambda\mu = 1. \tag{6}$$

The degenerate Bernoulli polynomials were subject of recent study by Adelberg [1], Howard [6], and Young [17,18] (these are sample references only). As noted in [18], the Bernoulli polynomials of the first kind satisfy $B_n(x) = \beta_n(0, x)$, since $(1 + \lambda t)^{\mu} \to e^t$ as $\lambda \to 0$. Similarly, the Bernoulli polynomials of the second kind satisfy $b_n(x) = \lim_{\lambda\to\infty} \lambda^{-n}\beta_n(\lambda, \lambda x)$, since $\lim_{\lambda\to\infty} \lambda((1 + t)^{1/\lambda} - 1) = \ln(1 + t)$. Both of these specializations are stated by Ustinov [16], using a different terminology, as explained below.

It appears that degenerate Bernoulli polynomials were rediscovered recently by Korobov [8] and Ustinov [16]. The *Korobov polynomials $K_n(x)$ of the first kind* are defined by the generating function

$$F_K(x, t) = \sum_{n=0}^{\infty} K_n(x)\frac{t^n}{n!} = \frac{pt(t + 1)^x}{(1 + t)^p - 1},$$

the Korobov numbers of the first kind K_n are given by $K_n := K_n(0)$. The terminology Korobov polynomial of the first kind is due to Ustinov [16], who completed the analogy with the Bernoulli polynomials by also introducing the *Korobov polynomials $k_n(x)$* of the second kind by the generating function

$$F_k(x, t) = \sum_{n=0}^{\infty} k_n(x)\frac{t^n}{n!} = \frac{t(1 + pt)^{x/p}}{(1 + pt)^{1/p} - 1},$$

the Korobov numbers of the second kind k_n are given by $k_n := k_n(0)$.

Proposition 1.2. *The Korobov polynomials of the first and second kind may be expressed in terms of degenerate Bernoulli polynomials* (*of the first order*) *as follows*:

$$K_n^{(p)}(x) = p^n\beta_n(1/p, x/p) \quad \text{and} \quad k_n^{(p)}(x) = \beta_n(p, x).$$

The proof is immediate from the definitions.

1.2. Progressively finite games and Sprague–Grundy numbers

We only need as much of the theory of progressively finite games as it is readily available in current undergraduate textbooks, a sample reference is [15, Chapter 11]. A progressively finite two-player game is a game with finitely many positions that must end after a finite number of moves. The positions of the game may be represented with vertices of a directed graph that contains no directed cycle nor infinite path, the edges represent valid moves. The two players take alternate turns to move along a directed edge to a next position, until one of them reaches a *winning position* with no edge going out: the player who moves into this position is declared a winner, the next player is unable to move.

The winning strategy for such a progressively finite game may be found by calculating the *Grundy number* (or Sprague–Grundy number) of each position. The Grundy numbers are defined recursively as follows.

- All winning positions have Grundy number 0.
- If the Grundy numbers on all successors of a position is known, then the Grundy number of the position is the least natural number that does not appear as the Grundy number of one of its successors.

The positions with Grundy number zero are called *kernel positions*. A player has a winning strategy exactly when he or she is allowed to start from a non-kernel position. In fact, by the definition of the labeling, there is always a way to move from a non-kernel position to a kernel position, whereas from the kernel one can only move to a non-kernel position. All winning positions are in the kernel, thus a player that keeps moving into the kernel, gives no chance to the opponent to win.

2. The original Bernoulli game

In this section we define and analyze a progressively finite game that turns out to be related to the Bernoulli numbers of the second kind. All other games related to (generalized) Bernoulli polynomials and numbers are generalizations of this game in one way or other. Thus we will refer to this game as the *original Bernoulli game*.

Definition 2.1. The positions in the original Bernoulli game are all pairs of words $(u_1 \cdots u_n, v_1 \cdots v_n)$ (where $n > 0$) such that

(i) the letters $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are positive integers;
(ii) for each $i \geqslant 1$ we have $1 \leqslant u_i, v_i \leqslant i$.

A *valid move* consists of replacing the pair $(u_1 \cdots u_n, v_1 \cdots v_n)$ with $(u_1 \cdots u_m, v_1 \cdots v_m)$ for some $m \geqslant 1$ satisfying $u_{m+1} \leqslant v_j$ for $j = m+1, \dots, n$.

In particular, $(u_1, v_1) = (1, 1)$ is a winning position, which may be reached in a single move from $(u_1 \cdots u_n, v_1 \cdots v_n)$ if and only if $u_2 \leqslant v_j$ holds for $j = 2, \dots, n$. We define the *rank* of a position $(u_1 \cdots u_n, v_1 \cdots v_n)$ to be the length n of the words $u_1 \cdots u_n$ and $v_1 \cdots v_n$. Each move decreases the rank of the position, thus the game is progressively finite.

Our main result is the following theorem.

Theorem 2.2. *For $n \geqslant 1$, the number κ_n of kernel positions of rank n in the Bernoulli game is given by*

$$\kappa_n = (-1)^{n-1}(n+1)!b_n,$$

where b_n is the nth Bernoulli number of the second kind.

Since there are $(n!)^2$ positions of rank n, Theorem 2.2 may be rephrased as follows.

Corollary 2.3. *Assume that the starting position of the Bernoulli game is selected at random among all positions of rank n (n is fixed), according to the uniform distribution. Then the probability that the game starts with a kernel position (and thus the second player has a winning strategy) is*

$$p_n = \frac{(-1)^{n-1}(n+1)b_n}{n!}.$$

Remark 2.4. Theorem 2.2 provides a combinatorial proof of the fact that the sign of b_n is $(-1)^{n-1}$. This is shown by analytic means in Jordan [7, p. 267], more recent references include [10, Theorem 2.1] (the Cauchy number $\mathcal{C}_n$ is equal to b_n) and a generalization by Young [18], right after Eq. (3.15). To our best knowledge, ours is the first combinatorial argument.

We prove Theorem 2.2 by proving three lemmas and a proposition first. The first lemma is trivial, but plays a crucial role in enumerating the kernel positions.

Lemma 2.5. *For each non-kernel position $(u_1 \cdots u_n, v_1 \cdots v_n)$ there is a unique kernel position $(u_1 \cdots u_m, v_1 \cdots v_m)$ that may be reached in a single move.*

Proof. Assume by way of contradiction that the positions $(u_1 \cdots u_m, v_1 \cdots v_m)$ and $(u_1 \cdots u_{m'}, v_1 \cdots v_{m'})$ are both kernel positions that are reachable from the same non-kernel position $(u_1 \cdots u_n, v_1 \cdots v_n)$ in a single move. Without loss of generality we may assume $m < m'$. Since $(u_1 \cdots u_m, v_1 \cdots v_m)$ is reachable from $(u_1 \cdots u_n, v_1 \cdots v_n)$ in a single move, by Definition 2.1 we must have $u_{m+1} \leqslant v_j$ for $j = m+1, \ldots, n$. But then, by restriction, we also have $u_{m+1} \leqslant v_j$ for $j = m+1, \ldots, m'$, thus there is a directed edge from the position $(u_1 \cdots u_{m'}, v_1 \cdots v_{m'})$ to the position $(u_1 \cdots u_m, v_1 \cdots v_m)$. This is a contradiction, since two kernel vertices cannot be connected by an edge. □

The following statement is stated as a separate lemma, because several of its instances will be used in the proofs of some subsequent theorems.

Lemma 2.6. *Let $d \geqslant 0$ and $m > 0$ be integers. The set of all words $z_1 z_2 \cdots z_m$ satisfying $1 \leqslant z_i \leqslant d + i$ for $i = 1, 2, \ldots, m$ and $z_1 < z_2, \ldots, z_m$ is in bijection with the set of those m-permutations $x_1 \cdots x_m$ of the set $\{1, 2, \ldots, d+m\}$ which satisfy $x_m = \min\{x_1, \ldots, x_m\}$.*

Proof. Set $x_1 := z_m$. Assuming that $x_1, \ldots, x_{j-1}$ are already defined, determine x_j as follows. Number the elements of the set $\{1, 2, \ldots, d+m\} \setminus \{x_1, \ldots, x_{j-1}\}$ from the least to the largest in increasing order. Let x_j be that element of $\{1, 2, \ldots, d+m\} \setminus \{x_1, \ldots, x_{j-1}\}$ whose number is z_{m+1-j}. In other words, x_j is the z_{m+1-j}th smallest element of $\{1, 2, \ldots, d+m\} \setminus \{x_1, \ldots, x_{j-1}\}$. The resulting word $x_1 \cdots x_m$ is an m-permutation of $\{1, 2, \ldots, d+m\}$. Conversely, each m-permutation $x_1 \cdots x_m$ of $\{1, 2, \ldots, d+m\}$ is encoded by a word $z_1 \cdots z_m$ satisfying $1 \leqslant z_i \leqslant d+i$ for $i = 1, 2, \ldots, m$. Clearly x_m is the least element of the set $\{x_1, \ldots, x_m\}$ if and only if each previous x_j has a higher number in what is left of $\{1, 2, \ldots, m+d\}$ at the moment when it is chosen. □

The last lemma shows that, for a fixed m and n satisfying $1 \leqslant m < n$, any position of rank m may be reached from the same number of positions of rank n in a single move.

Lemma 2.7. *Any position $(u_1 \cdots u_m, v_1 \cdots v_m)$ of rank m may be reached from exactly*

$$(n)_{n-m-1} \cdot (n+1)_{n-m+1} / (n-m+1)$$

positions of rank n in a single move.

Proof. The position $(u_1 \cdots u_m, v_1 \cdots v_m)$ is reachable from $(u_1 \cdots u_n, v_1 \cdots v_n)$ in a single move exactly when the criterion in Definition 2.1 is satisfied. Since each move is a restriction, $u_1, \ldots, u_m$ and $v_1, \ldots, v_m$ are already given. There is no requirement about $u_{m+2}, \ldots, u_n$ in the definition of a valid move, thus the number of ways to select these is $(m+2) \cdot (m+3) \cdots n = (n)_{n-m-1}$, by criterion (ii).

We are left with determining the number of ways to choose $u_{m+1}, v_{m+1}, v_{m+2}, \ldots v_n$ satisfying the criterion of a valid move. Set $z_1 := u_{m+1}$, $z_2 := v_{m+1} + 1$, …, $z_{n-m+1} := v_n + 1$. The condition for a valid move is equivalent to setting $1 \leqslant z_1 \leqslant m+1$, and $z_1 < z_j \leqslant m+j$ for $j \geqslant 2$. By Lemma 2.6 the set of all words $z_1 \cdots z_m$ is in bijection with the set of all $(n-m+1)$ permutations of the set $\{1, 2, \ldots, m+n-m+1\} = \{1, 2, \ldots, n+1\}$ which end with the least letter. The number of such permutations is clearly $(n+1)_{n-m+1}/(n-m+1)$. □

Using Lemmas 2.5 and 2.7 we may prove the following recursion formula for the number κ_n of kernel positions of rank n.

Proposition 2.8. *The numbers κ_n of kernel positions of rank n in the original Bernoulli game satisfy the equation*

$$(n!)^2 = \kappa_n + \sum_{m=1}^{n-1} \kappa_m \cdot (n)_{n-m-1} \cdot \frac{(n+1)_{n-m+1}}{n-m+1}. \tag{7}$$

Proof. The number $(n!)^2$ on the left-hand side of the equation is the total number of positions of rank n. Among these $(n!)^2$ positions there are κ_n kernel positions. By Lemma 2.5 there is a unique kernel position reachable from each non-kernel position of rank n. By Lemma 2.7, for each $m \in \{1, \dots, n-1\}$, any kernel position of rank m may be reached from exactly $(n)_{n-m-1} \cdot (n+1)_{n-m+1}/(n-m+1)$ non-kernel positions of rank n. $\Box$

Proof of Theorem 2.2. Dividing both sides by $n!(n+1)!$ in (7) yields

$$\frac{1}{n+1} = \frac{\kappa_n}{n!(n+1)!} + \sum_{m=1}^{n-1} \frac{\kappa_m}{m!(m+1)!} \cdot \frac{1}{n-m+1}. \tag{8}$$

Introducing

$$\tilde{b}_0 := 1 \quad \text{and} \quad \tilde{b}_j := (-1)^{j-1} \cdot \frac{\kappa_j}{j!(j+1)!} \quad \text{for } j \geqslant 1, \tag{9}$$

we may rewrite (8) as

$$\frac{\tilde{b}_0}{n+1} = (-1)^{n-1}\tilde{b}_n + \sum_{m=1}^{n-1} (-1)^{m-1}\tilde{b}_m \cdot \frac{1}{n-m+1},$$

which is equivalent to

$$\sum_{m=0}^{n} (-1)^m \tilde{b}_m \cdot \frac{1}{n-m+1} = 0.$$

This last equation is equivalent to the recursion formula (5), one only needs to replace n with $n+1$. The numbers $\tilde{b}_n$ satisfy therefore the same initial condition and recursion formula as Jordan's Bernoulli numbers of the second kind $\psi_n(0)$. The equation $\tilde{b}_n = \psi_n(0) = b_n/n!$ for all n follows by induction on n. $\Box$

3. Enumeration by kernel positions

In enumerating the kernel positions in the original Bernoulli game we used the following fundamental properties of the game:

(i) The set of positions has a rank function associated to it, which decreases after each move.
(ii) From each non-kernel position exactly one kernel position could be reached in a single move.
(iii) Each position of rank $m < n$ could be reached from the same number of positions of rank n in a single move.

Observe that the rank function is not directly related to the partial order that arises by taking the transitive closure of the graph of valid moves, but to the partial order induced by the restriction operation. This second partial order is a proper extension of the first.

Based on these observations we may generalize the original Bernoulli game to partially ordered sets as follows. Recall that a *rank function* on a partially ordered set P is a function $\rho : P \to \mathbb{Z}$ satisfying $\rho(x) \leqslant \rho(y)$ whenever $x \leqslant y$, and $\rho(y) = \rho(x) + 1$ whenever y covers x. If the poset has a unique minimum element $\hat{0}$ it is usual to require $\rho(\hat{0}) = 0$.

Definition 3.1. Let P be a (possibly infinite) partially ordered set with a unique minimum element $\hat{0}$, a rank function ρ satisfying $\rho(\hat{0}) = 0$, and assume that for each $n \in \mathbb{N}$ the set P_n of elements of rank n in P is finite. We say that a function $M : P \to \mathcal{P}(P)$ assigning to each $x \in P$ a subset of P *induces a Bernoulli type game* if it satisfies the following criteria:

(i) For each $x \in P$, $M(x)$ is a (possibly empty) chain in the half-open interval $[\hat{0}, x)$. (In particular, $M(\hat{0}) = \emptyset$.)
(ii) If $y_1, y_2 \in M(x)$ and $y_1 < y_2$ then $y_1 \in M(y_2)$.
(iii) For all $m < n$ there is a number $\gamma_{m,n}$ such that each $y \in P_m$ belongs to $M(x)$ for exactly $\gamma_{m,n}$ elements $x \in P_n$. (In other words, $|M^{-1}(y) \cap P_n| = \gamma_{m,n}$ for all $y \in P$ of rank m.)

We define the *game induced by* (P, M) as the two-player game whose set of positions is P and whose valid moves consist of moving from $x \in P$ to any element of $M(x)$. The winning positions are the ones satisfying $M(x) = \emptyset$. Since each valid move decreases the rank, the game is progressively finite. Criterion (ii) allows us to state the following analog of Lemma 2.5.

Corollary 3.2. *In a Bernoulli type game, for each non-kernel position x there is a unique kernel position $y \in M(x)$.*

Introducing κ_n for the number of kernel positions of rank n, and using criterion (iii) we may state the following generalization of Proposition 2.8

Proposition 3.3. *The numbers κ_n of kernel positions of rank n in a Bernoulli type game satisfy the equation*

$$|P_n| = \kappa_n + \sum_{m=0}^{n-1} \kappa_m \cdot \gamma_{m,n}. \tag{10}$$

Given $\kappa_0 = 1$, the numbers $|P_n|$ and $\gamma_{m,n}$, Eq. (10) may be used to recursively calculate the numbers κ_n. It is to be expected that whenever we start with a pair (P, M) that has a "nice structure," we get "interesting" sequences $\{\kappa_n\}$.

Remark 3.4. The positions of the original Bernoulli game are partially ordered by the relation $(u_1 \cdots u_m, v_1 \cdots v_m) < (u_1 \cdots u_n, v_1 \cdots v_n)$ for all $m < n$ (taking initial segments), and the unique minimum element of this partially ordered set is the pair $(1, 1)$ the only element of rank 1. For the sake of consistency with the theory presented in this section we may add an extra minimum element $\hat{0} := (\emptyset, \emptyset)$ at rank zero, or accept the fact that the rank function in the original Bernoulli game is shifted by one.

Definition 3.1 forces the following properties of the operation M.

Proposition 3.5. *Assume the pair (P, M) induces a Bernoulli type game. Then the set $M^*(x)$ of all elements that are reachable from x in any number of moves (including not moving at all) is a chain in the closed interval $[\hat{0}, x]$. Furthermore, if $y_1, y_2 \in M^*(x) \setminus \{x\}$ and $y_1 < y_2$ then $y_1 \in M^*(y_2)$.*

Proof. Clearly all elements reachable from x are less than or equal to x. We prove by induction on $(\rho(x) - \rho(z_1)) + (\rho(x) - \rho(z_2))$ the following statement: for any $x \in P$, any pair of elements $z_1, z_2 \in M^*(x) \setminus \{x\}$ is comparable, and the lesser one of z_1 and z_2 is reachable from the larger one.

If both z_1 and z_2 belong to $M(x)$ then the statement follows from conditions (i) and (ii) in Definition 3.1. Thus we may assume that at least one of z_1 and z_2 is not in $M(x)$. Without loss of generality we may assume $z_1 \notin M(x)$. This means that there is an $x' \in M(x)$ such that $z_1 \in M^*(x')$. Now x' and z_2 both belong to $M^*(x)$ and we have

$$\big(\rho(x) - \rho(x')\big) + \big(\rho(x) - \rho(z_2)\big) < \big(\rho(x) - \rho(z_1)\big) + \big(\rho(x) - \rho(z_2)\big).$$

By our induction hypothesis for $x \in P$, $x', z_2 \in M^*(x)$ we obtain that x' and z_2 must be comparable and the lesser of x' and z_2 is reachable from the larger one. If $x' > z_2$ then $z_2 \in M^*(x')$. Since we also have $z_1 \in M^*(x')$ the statement for z_1 and z_2 follows from the induction hypothesis for $x' \in P$ and $z_1, z_2 \in M^*(x')$, since we have

$$\big(\rho(x') - \rho(z_1)\big) + \big(\rho(x') - \rho(z_2)\big) < \big(\rho(x) - \rho(z_1)\big) + \big(\rho(x) - \rho(z_2)\big).$$

If $x' < z_2$, then $x' \in M^*(z_2)$. Since we also have $z_1 \in M^*(x')$, we obtain $z_1 \in M^*(z_2)$, which also implies $z_1 < z_2$. $\square$

As a consequence of Proposition 3.5, once we have selected the starting position $x \in P$ in a Bernoulli type game induced by some (P, M), the game is played out on a single chain $M^*(x) = \{x_1, \dots, x_s\}$. Here we may assume $\hat{0} \leqslant x_1 < x_2 < \cdots \leqslant x_s = x$ but we cannot assume any relation between the index j of x_j and the rank $\rho(x_j)$. By the second half of Proposition 3.5 we have $M^*(x_j) = \{x_1, \dots, x_j\}$ for $j = 1, 2, \dots, s$.

Lemma 3.6. *Assume $y_1, y_2, y_3 \in M^*(x)$ satisfy $y_1 < y_2 < y_3$ and $y_1 \in M(y_3)$. Then $y_1 \in M(y_2)$.*

Proof. As noted before the lemma, both y_1 and y_2 belong to $M^*(y_3)$. We prove the claim by induction on $(\rho(y_3) - \rho(y_1)) + (\rho(y_3) - \rho(y_2))$. If $y_2 \in M(y_3)$ then $y_1 \in M(y_2)$ follows from condition (ii) in Definition 3.1. If $y_2 \in M^*(y_3) \setminus M(y_3)$ then there is a $y \in M^*(x)$ strictly between y_2 and y_3 such that $y \in M(y_3)$ and $y_2 \in M^*(y)$ hold. By condition (ii) in Definition 3.1, $y, y_1 \in M(y_3)$ and $y_1 < y$ imply $y_1 \in M(y)$. Now we may apply the induction hypothesis to $y_1 < y_2 < y$. $\square$

As a consequence of Lemma 3.6, to each $x_i \in M^*(x) \setminus \{x\}$ we may associate a unique $\nu(i) > i$ such that x_i belongs to $M(x_{i+1}), M(x_{i+2}), \dots, M(x_{\nu(i)})$ and to no other $M(x_k)$. The function $\nu : \{1, \dots, s-1\} \to \{1, \dots, s\}$ satisfies $\nu(j) > j$ for all j. This function completely determines the restriction of M to $M^*(x)$. We have just shown that in each Bernoulli type game, after selecting the starting position x we are left with a game that is equivalent to the following.

Definition 3.7. For a positive integer s, let $\nu : \{1, 2, \dots, s-1\} \to \{1, 2, \dots, s\}$ be a function satisfying $\nu(j) > j$ for all j. The *Bernoulli game induced by* ν is the two-player game whose positions are $1, \dots, s$, and whose set of valid moves consist of all moves (j, i) satisfying $i < j \leqslant \nu(i)$. The players take alternate turns, the player unable to move loses. In this game we insist that the starting position must be s.

Theorem 3.8. *Given any position $x \in P$ in a Bernoulli type game induced by (P, M), the game started at x is isomorphic to a Bernoulli game induced by some $\nu : \{1, 2, \dots, s-1\} \to \{1, 2, \dots, s\}$ satisfying $\nu(j) > j$ for all j. Conversely, for any function $\nu : \{1, 2, \dots, s-1\} \to \{1, 2, \dots, s\}$ satisfying $\nu(j) > j$ for all j, there is a pair (P, M) and a position $x \in P$ such that a game started at x is isomorphic to the Bernoulli game induced by ν.*

Proof. The first half of the statement was shown before Definition 3.7. Assume we are given a function $\nu : \{1, 2, \dots, s-1\} \to \{1, 2, \dots, s\}$ satisfying $\nu(j) > j$ for all j. Let P be the set $\{1, \dots, s\}$, linearly ordered by the natural order, and the operation M given by $M(i) = \{j < i\colon\ i < \nu(j)\}$. It is easy to see that the pair (P, M) satisfies the criteria given in Definition 3.1. In fact, for each $i \in P$ the set $M(i)$ is a chain in $\{1, \dots, i-1\}$ so condition (i) is satisfied. Assume $j_1, j_2 \in M(i)$ and $j_1 < j_2$. Since $j_1 \in M(i)$, we have $i \leqslant \nu(j_1)$. Since $j_2 \in M(i)$, we have $j_2 < i$. Thus $j_2 < \nu(j_1)$ and $j_1 < j_2$ hold, implying $j_1 \in M(j_2)$ and the validity of condition (ii). At last, condition (iii) is trivially satisfied since we have exactly one element at each rank. Finally let us note that $M^*(s) = \{1, \dots, s\}$. In fact, since $\nu(j) > j$, every $j < s$ is reachable from at least one larger number, repeated application of this observation leads to proving $j \in M^*(s)$. $\square$

Remark 3.9. Theorem 3.8 completely characterizes the two player games we are left to play, once the starting position is selected in any Bernoulli type game, induced by some (P, M). The isomorphism of games does not extend, however, to preserving the rank functions: the set of ranks $\{\rho(x_1), \dots, \rho(x_s)\}$ associated to some $M^*(x) = \{x_1, \dots, x_s\}$ may be any set of non-negative integers, whereas the elements of the set $\{1, \dots, s\}$, ordered by the natural order may be found at consecutive ranks. This must be considered if anyone ever tries to use Theorem 3.8 to calculate the number of kernel positions of a given rank in any Bernoulli type game.

As a consequence of Theorem 3.8, it suffices to find the kernel positions for a Bernoulli game induced by some function ν satisfying Definition 3.7 to understand how to play in any Bernoulli type game, even if counting kernel positions by finding the strategy remains elusive, due to the difficulty mentioned in Remark 3.9.

Theorem 3.10. *Let $\nu : \{1, \dots, s-1\} \to \{1, \dots, s\}$ be a function satisfying $\nu(j) > j$ for all j. Let $\hat{\nu} : \{1, \dots, s-1\} \to \{1, \dots, s\}$ be the function defined by $\hat{\nu}(j) := \nu(j) + 1$. Then the set of kernel positions of the Bernoulli type game induced by ν is*

$$\bigl\{1, \hat{\nu}(1), \hat{\nu}^2(1), \dots, \hat{\nu}^{s-1}(1)\bigr\} \cap \{1, 2, \dots, s\}.$$

Here $\hat{\nu}^j$ is the jth compositional power of $\hat{\nu}$.

Proof. 1 is the only winning position in the game, and it is clearly a kernel position. Any $j \in \{2, \dots, \nu(1)\}$ satisfies $1 \in M(j)$ and is thus a non-kernel position. The element $\hat{\nu}(1) = \nu(1) + 1$ is the least element from which 1 is not reachable and all elements of $M(\hat{\nu}(1)) \neq \emptyset$ belong to $\{2, \dots, \nu(1)\}$. Thus $\hat{\nu}(1)$ is a kernel position. We prove by induction on $k \geqslant 1$ that the only kernel positions in the set $\{1, 2, \dots, \hat{\nu}^k(1)\}$, are 1, $\hat{\nu}(1)$, $\hat{\nu}^2(1)$, $\dots$, $\hat{\nu}^k(1)$. We have just shown the claim for $k = 1$. Assume that claim is true for some $k > 0$. Any $j \in \{\hat{\nu}^k(1) + 1, \hat{\nu}^k(1) + 2, \dots, \nu(\hat{\nu}^k(1))\}$ satisfies $\hat{\nu}^k(1) \in M(j)$ and is thus a non-kernel position. The element $\nu(\hat{\nu}^k(1)) + 1 = \hat{\nu}^{k+1}(1)$ is the least element above $\hat{\nu}^k(1)$ from which $\hat{\nu}^k(1)$ is not reachable in a single step. We are left to show that this is a kernel position. Assume by way of contradiction that it is not. Then at least one kernel position is reachable from $\hat{\nu}^{k+1}(1)$ in a single step. By our induction hypothesis, such a kernel position is of the form $\hat{\nu}^m(1)$ for some $0 \leqslant m < k$. (We set $m = 0$ for $\hat{\nu}^0(1) := 1$.) But then $\hat{\nu}^{k+1}(1)$ is less than equal to $\nu(\hat{\nu}^m(1)) = \hat{\nu}^{m+1}(1) - 1 \leqslant \hat{\nu}^k(1) - 1$, in contradiction with the fact that $\hat{\nu}(j) > j$ holds for all j. $\square$

4. Polynomial Bernoulli games

In this section we construct two variants of the Bernoulli game that are related to negative integer substitutions into the Bernoulli polynomials of the first and second kind. We begin with the Bernoulli polynomials of the second kind, since the game introduced for them is closer to the original Bernoulli game.

The defining Eq. (3) for the Bernoulli polynomials of the second kind may be rewritten as

$$\sum_{n=0}^{\infty} \frac{b_n(x)}{n!} t^n \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n} = t \sum_{n=0}^{\infty} \binom{x}{n} t^n.$$

Comparing the coefficients of t yields $b_0(x) = 1$, comparing coefficients of t^{n+1} for $n \geqslant 0$ yields

$$\frac{b_n(x)}{n!} + \sum_{m=0}^{n-1} \frac{b_m(x)}{m!} \cdot \frac{(-1)^{n-m}}{n+1-m} = \binom{x}{n}. \tag{11}$$

Using the identity

$$\binom{x}{n} = (-1)^n \binom{-x+n-1}{n}$$

and replacing x with $-x$ we may rewrite (11) as

$$\frac{b_n(-x)}{n!} + \sum_{m=0}^{n-1} \frac{b_m(-x)}{m!} \cdot \frac{(-1)^{n-m}}{n+1-m} = (-1)^n \binom{x+n-1}{n}.$$

Multiplying both sides by $(-1)^n n!(n+1)!$ we obtain

$$(-1)^n (n+1)! b_n(-x) + \sum_{m=0}^{n-1} (-1)^m \frac{b_m(-x)}{m!} \frac{n!(n+1)!}{(n+1-m)} = n!(n+1)! \binom{x+n-1}{n}.$$

Introducing $\kappa_n := (-1)^n (n+1)! b_n(-x)$ for $n \geqslant 0$ we may rewrite the last equation as

$$\kappa_n + \sum_{m=0}^{n-1} \kappa_m (n)_{n-m+1} \frac{(n+1)_{n+1-m}}{n+1-m} = n!(n+1)! \binom{x+n-1}{n}. \tag{12}$$

This recursion formula inspires the definition of the following game.

Definition 4.1. Let $x \geqslant 1$ be a positive integer. The positions in the *polynomial Bernoulli game of the second kind, indexed with* x are triplets of words $(u_1 \cdots u_n, v_1 \cdots v_n, w_1 \cdots w_n)$ for $n \geqslant 0$ such that

(i) $1 \leqslant u_i \leqslant i$ holds for $i \geqslant 1$;
(ii) $1 \leqslant v_i \leqslant i+1$ holds for $i \geqslant 1$;
(iii) $1 \leqslant w_i \leqslant x$ and $w_i \leqslant w_{i+1}$ hold for $i \geqslant 1$.

A *valid move* consists of replacing $(u_1 \cdots u_n, v_1 \cdots v_n, w_1 \cdots w_n)$ with $(u_1 \cdots u_m, v_1 \cdots v_m, w_1 \cdots w_m)$ for some $m \geqslant 0$ satisfying the following conditions:

(a) $u_{m+1} < v_j$ for $j = m+1, \dots, n$;
(b) $w_{m+1} = w_{m+2} = \cdots = w_n = x$.

The restrictions on the words $u_1 \cdots u_n$ and $v_1 \cdots v_n$ are similar to the ones in the original Bernoulli game, and condition (a) is similar to the condition on the valid move in that game. Major differences between the original Bernoulli game and this extended version include allowing a triplet of empty words as a valid position, and requiring strict inequality in condition (a). In analogy to the original Bernoulli game, one may define a partial order on the set of positions by taking initial segments of the words involved, and a rank function by taking the common length of the words in the triplet. It is easy to verify that the resulting partially ordered set P and the function M induced by the definition of a valid move above satisfies the criteria given in Definition 3.1.

Theorem 4.2. *For a positive integer $x \geqslant 1$, the number of kernel positions of rank n in the polynomial Bernoulli game of the second kind, indexed with x is*

$$\kappa_n := (-1)^n (n+1)! b_n(-x).$$

Proof. We prove the theorem by showing that the number κ_n of kernel positions of rank n satisfy the recursion formula (12). Let us observe first that the expression on the right-hand side is the number of all positions of rank n. In fact, the number of words $u_1 \cdots u_n$ is $n!$, the number of words $v_1 \cdots v_n$ is $(n+1)!$, finally, the number of words $w_1 \cdots w_n$ is the number of n-combinations (with repetitions) of an x-element set, i.e., $\binom{x+n-1}{n}$. There are exactly κ_n kernel positions of rank n, and for any other position there is a unique $m<n$ and a unique kernel position of rank m that is reachable from it. Thus it is sufficient to show that any position of rank $m<n$ is reachable from exactly

$$\gamma_{m,n}=(n)_{n-m+1}\frac{(n+1)_{n+1-m}}{n+1-m}$$

positions of rank n, and the recursion formula (12) will follow from Proposition 3.3. Here $u_1,\dots,u_m$, $v_1,\dots,v_m$ and $w_1\dots w_m$ are already given and, by condition (b), we must have $w_{m+1}=\cdots=w_n=x$. There is no other condition on $u_{m+2},\dots,u_n$ than what is given in (i) thus we have $n!/(m+1)!$ ways to choose these letters. We are left to show that we may select the numbers $u_{m+1},v_{m+1},\dots,v_n$ in exactly $(n+1)_{n+1-m}/(n+1-m)$ ways such that they satisfy the condition (a) in the definition of a valid move. By Lemma 2.6, the set of all sequences $u_{m+1},v_{m+1},\dots,v_n$ satisfying (i), (ii) and (b) is in bijection with the set of those $(n-m+1)$-permutations of $\{1,2,\dots,n+1\}$ which end with the smallest letter. The number of such $(n-m+1)$-permutations is $(n+1)_{n+1-m}/(n+1-m)$. □

Corollary 4.3. *Assume that the starting position of the polynomial Bernoulli game of the second kind, indexed with x is selected at random among all positions of rank n (n is fixed), according to the uniform distribution. Then the probability that the game starts with a kernel position is*

$$p_n=\frac{(-1)^n(n+1)!b_n(-x)}{n!(n+1)!\binom{x+n-1}{n}}=\frac{(-1)^nb_n(-x)}{(x+n-1)_n}.$$

The polynomial Bernoulli game of the second kind has the property that valid moves may occur only to positions of rank m satisfying $w_{m+1}=\cdots=w_n=x$. In particular, a position $(u_1\cdots u_d,v_1\cdots v_d,w_1\cdots w_d)$ is simply not reachable from any position of higher rank with $w_{d+j}\neq x$ for some $j>0$, and neither is any initial segment from it. For any $n>d$ satisfying $w_{d+j}\neq x$ for some $j>0$, we may reduce the analysis of a polynomial Bernoulli game with starting position $(u_1\cdots u_n,v_1\cdots v_n,w_1\cdots w_n)$ to considering the game on the subwords $(u_{d+1}\cdots u_n,v_{d+1}\cdots v_n,w_{d+1}\cdots w_n)$. This observation suggests considering the following shifted version of the polynomial Bernoulli game of the second kind.

Definition 4.4. Let $d\geqslant 0$ be a non-negative integer and $x\geqslant 1$ be a positive integer. We define the *d-shifted* polynomial Bernoulli game of the second kind, indexed with $x>0$, by modifying the definition of the polynomial Bernoulli game of the second kind, indexed with x, as follows:

(i) replace the condition $1\leqslant u_i\leqslant i$ by $1\leqslant u_i\leqslant i+d$;
(ii) replace the condition $1\leqslant v_i\leqslant i+1$ by $1\leqslant v_i\leqslant i+d+1$.

The rest of the definition of a valid position and a valid move remains unchanged.

In particular, the 0-shifted polynomial Bernoulli games of the second kind are the same as the ones defined before. In analogy to the proof of Theorem 4.2 it is easy to show the following.

Proposition 4.5. *Given $d\geqslant 0$ and $x>1$, the numbers κ_n of kernel positions of rank n in the d-shifted polynomial Bernoulli game of the second kind, indexed with x, satisfy the recursion formula*

$$\kappa_n+\sum_{m=0}^{n-1}\kappa_m(n+d)_{n-m+1}\frac{(n+d+1)_{n+1-m}}{n+1-m}=\frac{(n+d)!(n+d+1)!}{d!(d+1)!}\binom{x+n-1}{n}. \tag{13}$$

Multiplying both sides with $d!(d+1)!n!(n+1)!/((n+d)!(n+d+1)!)$ in (13) and introducing

$$\tilde{\kappa}_n:=\frac{d!(d+1)!n!(n+1)!\kappa_n}{(n+d)!(n+d+1)!}$$

we may rewrite (13) as

$$\tilde{\kappa}_n+\sum_{m=0}^{n-1}\tilde{\kappa}_m(n)_{n-m+1}\frac{(n+1)_{n+1-m}}{n+1-m}=n!(n+1)!\binom{x+n-1}{n},$$

which is exactly the recursion formula for the numbers κ_n associated to the (unshifted) polynomial Bernoulli game of the second kind, indexed with x. Thus we obtain the following.

Theorem 4.6. *For a positive integer $x \geqslant 1$, and a non-negative integer d, the number of kernel positions of rank n in the d-shifted polynomial Bernoulli game of the second kind indexed with x is*

$$\kappa_n := \frac{(-1)^n(n+d)!(n+d+1)!b_n(-x)}{d!(d+1)!n!}.$$

Consider now the unshifted polynomial Bernoulli game of the second kind, indexed with x, and a starting position $(u_1\cdots u_n, v_1\cdots v_n, w_1\cdots w_n)$. Given any $y \leqslant x$, there is a unique $d \geqslant 0$ such that $w_d \leqslant y$ and $w_{d+1} \geqslant y+1$. Deciding the outcome of the game is equivalent to deciding the outcome of a d-shifted polynomial Bernoulli game of the second kind, indexed with $x-y$, on the triplet $(u_{d+1}\cdots u_n, v_{d+1}\cdots v_n, w_{d+1}\cdots w_n)$. For a fixed $(u_1\cdots u_d, v_1\cdots v_d, w_1\cdots w_d)$, by Theorem 4.6, there are

$$\frac{(-1)^{n-d}n!(n+1)!b_{n-d}(-x+y)}{(n-d)!}$$

ways to find a triplet $(u_{d+1}\cdots u_n, v_{d+1}\cdots v_n, w_{d+1}\cdots w_n)$ satisfying $w_{d+1} \geqslant y+1$, representing a kernel position. On the other hand, a triplet $(u_1\cdots u_d, v_1\cdots v_d, w_1\cdots w_d)$, satisfying $w_d \leqslant y$ may be chosen $d!(d+1)!\binom{y+d-1}{d}$ ways. Summing over d allows to enumerate the kernel positions of rank n for the unshifted polynomial Bernoulli game of the second kind, indexed with x. Thus we obtain

$$(-1)^n(n+1)!b_n(-x)=\sum_{d=0}^{n}d!(d+1)!\binom{y+d-1}{d}\frac{(-1)^{n-d}n!(n+1)!b_{n-d}(-x+y)}{d!(d+1)!(n-d)!}.$$

Dividing both sides by $n!(n+1)!$ and simplifying yields the following shifting formula, which takes its most compact form for Jordan's [7] Bernoulli polynomials of the second kind:

$$(-1)^n\psi_n(-x)=\sum_{d=0}^{n}\binom{y+d-1}{d}(-1)^{n-d}\psi_{n-d}(-x+y).$$

Replacing x with $-x$, y with $-y$, and multiplying both sides by $(-1)^n$ yields

$$\psi_n(x)=\sum_{d=0}^{n}\binom{y}{d}\psi_{n-d}(x-y)\quad\text{for all } y \text{ satisfying } 0\leqslant y\leqslant x. \tag{14}$$

This shifting formula may be easily verified directly, from the definitions.

It appears that the properties of the Bernoulli polynomials of the first kind are not conducive to follow through a direct analogue of the above reasoning. On the other hand, the following plausible analogue of the polynomial Bernoulli game of the second kind is related to the Bernoulli polynomials of the first kind, albeit in a less direct way.

Definition 4.7. Let $x \geqslant 1$ be a positive integer. The positions in the *polynomial Bernoulli game of the first kind, indexed with* x are triplets of words $(u_1\cdots u_n, v_1\cdots v_n, w_1\cdots w_n)$ for $n \geqslant 0$ such that

(i) $1 \leqslant u_i \leqslant x$ holds for $i \geqslant 1$;
(ii) $1 \leqslant v_i \leqslant x$ holds for $i \geqslant 1$;
(iii) $1 \leqslant w_i \leqslant i$ holds for $i \geqslant 1$.

A *valid move* consists of replacing $(u_1 \cdots u_n, v_1 \cdots v_n, w_1 \cdots w_n)$ with $(u_1 \cdots u_m, v_1 \cdots v_m, w_1 \cdots w_m)$ for some $m \geqslant 0$ satisfying the following conditions:

(a) $u_{m+1} < v_j$ for $j = m+1, \ldots, n$;
(b) $w_{m+1} < w_{m+2} < \cdots < w_n$.

Again we may define a partial order on the set of positions by taking initial segments of the words involved, and a rank function by taking the common length of the words in the pair. The resulting partially ordered set P and the function M induced by the definition of a valid move above satisfies the criteria given in Definition 3.1.

Proposition 4.8. *The number κ_n of kernel positions in the polynomial Bernoulli game of the first kind satisfies*

$$n!x^{2n} = \kappa_n + \sum_{m=0}^{n-1} \kappa_m x^{n-m-1} \binom{n}{n-m} \frac{1}{n-m+1} \big(B_{n-m+1}(x) - B_{n-m+1}(0)\big) \tag{15}$$

for $n \geqslant 0$.

Proof. The expression on the left-hand side is clearly the number of all positions of rank n. There are exactly κ_n kernel positions of rank n, and for any other position there is a unique $m < n$ and a unique kernel position of rank m that is reachable from it. Thus it is sufficient to show that any position of rank $m < n$ is reachable from exactly

$$\gamma_{m,n} = x^{n-m-1} \binom{n}{n-m} \frac{1}{n-m+1} \big(B_{n-m+1}(x) - B_{n-m+1}(0)\big)$$

positions of rank n. Here $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ and $w_1 \ldots w_m$ are already given. There is no other condition on $u_{m+2}, \ldots, u_n$ than what is given in (i) thus we have x^{n-m-1} ways to choose these letters. By condition (b), there are $\binom{n}{n-m}$ ways to choose $1 \leqslant w_{m+1} < \cdots < w_n \leqslant n$. We are left to show that we may select the numbers $u_{m+1}, v_{m+1}, \ldots, v_n$ in exactly $(B_{n-m+1}(x) - B_{n-m+1}(0))/(n-m+1)$ ways such that they satisfy the condition (a) in the definition of a valid move. Once we select the value of u_{m+1} to be u, each of $v_{m+1}, \ldots, v_n$ may take one of $x - u$ values, independently. Thus the number of ways to select these entries is

$$\sum_{u=1}^{x} (x-u)^{n-m} = \sum_{j=1}^{x-1} j^{n-m} = \frac{1}{n-m+1} \big(B_{n-m+1}(x) - B_{n-m+1}(0)\big)$$

by (2). $\square$

Since $B_1(x) - B_1(0) = x$, we may rewrite (15) as

$$n!x^{2n} = \sum_{m=0}^{n} \kappa_m x^{n-m-1} \binom{n}{n-m} \frac{1}{n-m+1} \big(B_{n-m+1}(x) - B_{n-m+1}(0)\big).$$

Dividing both sides by $n!x^{n-1}$, we obtain

$$x^{n+1} = \sum_{m=0}^{n} \frac{\kappa_m}{m!x^m} \frac{(B_{n-m+1}(x) - B_{n-m+1}(0))}{(n-m+1)!}.$$

Multiplying both sides by t^{n+1}, and summing over all non-negative values of n yields

$$\frac{xt}{1-xt} = \sum_{n=0}^{\infty} \frac{\kappa_n}{n!x^n} t^n \cdot \sum_{n=1}^{\infty} \frac{B_n(x) - B_n(0)}{n!} t^n.$$

Using (1) we obtain

$$\frac{xt}{1-xt} = \sum_{n=0}^{\infty} \frac{\kappa_n}{n!x^n} t^n \cdot \frac{t(e^{xt}-1)}{e^t - 1}$$

which yields

$$\sum_{n=0}^{\infty} \frac{\kappa_n}{n!x^n} t^n = \frac{x(e^t-1)}{(1-xt)(e^{xt}-1)}.$$

Substituting t/x into t yields

$$\sum_{n=0}^{\infty} \frac{\kappa_n}{n!x^{2n}} t^n = \frac{x}{1-t} \cdot \frac{e^{t/x}-1}{e^t-1} = \frac{x}{t(1-t)} \sum_{n=1}^{\infty} \frac{B_n(1/x) - B_n(0)}{n!} t^n.$$

By comparing the coefficients of t^n on both sides, we obtain the following theorem.

Theorem 4.9. *The number of kernel positions of rank n in the polynomial Bernoulli game of the first kind, indexed with x, is given by*

$$\kappa_n := x^{2n+1} n! \cdot \sum_{m=1}^{n+1} \frac{B_m(1/x) - B_m(0)}{m!} \quad \textit{for } n \geqslant 0.$$

It is worth noting that Theorem 4.9 takes a simpler form in terms of Jordan's [7] Bernoulli polynomials of the first kind, namely

$$\kappa_n = x^{2n+1} n! \cdot \sum_{m=1}^{n+1} \bigl(\phi_m(1/x) - \phi_m(0)\bigr) \quad \text{for } n \geqslant 0.$$

5. Scoring several rounds of a Bernoulli game

In this section we consider several rounds of the original or the polynomial Bernoulli games of the second kind, played by two players who keep scores. We assume that both players have read Section 3 and know how to find the winning strategy in any Bernoulli type game. The question, which player wins a round depends only on the choice of the starting position, which in round n will be selected among the positions of rank n according to the uniform distribution. We then instruct the loser of the round to pay the winner a certain amount. Using the wealth of results on the Bernoulli numbers and polynomials of the second kind in the literature, we are able to calculate the expected gain of a player in many situations. To assure a seamless transition to these formulas, we make the following definitions.

Definition 5.1. Let $c_0, c_1, \ldots, c_n, \ldots$ be a sequence of real numbers. The *original Bernoulli game scored by* $\{c_n\}_{n=0}^{\infty}$ is a sequence of rounds played by two players, A and B, according to the following rules.

- In round 0, player A pays player B c_0 dollars. (A receives $-c_0$ dollars from B, if $c_0 < 0$.)
- For each $n > 0$ such that $c_n \neq 0$, the two players play a round of the original Bernoulli game (round n), starting with a random position of rank n. The starting position is selected according to the uniform distribution.
- If $c_n > 0$ then A begins round n, otherwise B begins the round.
- If the player who started the round wins, no money is paid, otherwise the first player pays the second player $|c_n|$ dollars at the end of the round.

The *polynomial Bernoulli game of the second kind, indexed with x, scored by* $\{c_n\}$ is defined the same way, except in round n a polynomial Bernoulli game indexed with x is played, starting with a random starting position of rank n, selected by the uniform distribution.

As an easy consequence of the definitions, we obtain the following.

Proposition 5.2. *Consider the original, or a polynomial Bernoulli game, scored by* $\{c_n\}_{n=0}^{\infty}$. *Let* p_n *be the probability that the second player has a winning strategy in round* $n > 0$, *and set* $p_0 := 1$. *Assume that* $\sum_{n=0}^{\infty} c_n \cdot p_n$ *converges. Then the expected gain of player B is* $\sum_{n=0}^{\infty} c_n \cdot p_n$.

Using Corollaries 2.3 and 4.3, we may translate $\sum_{n=0}^{\infty} c_n \cdot p_n$ into a combination of Bernoulli numbers or Bernoulli polynomials. The value of many such expressions is known in the literature, and now we have a random process model for all problems that motivated the calculation of those expressions. The first class of examples is related to numerical integration.

The Bernoulli numbers of the second kind are intimately related to integrating a function given by its *Newton expansion*. The Newton expansion of a polynomial $f(x)$ is the sum

$$f(x) = f(0) + \Delta f(0)\binom{x}{1} + \Delta^2 f(0)\binom{x}{2} + \cdots + \Delta^n f(0)\binom{x}{n} + \cdots$$

in which only finitely many terms are not zero. The definition may be extended to other continuous functions, but then questions of convergence will arise.

Jordan's Bernoulli numbers of the second kind satisfy $\psi_n(0) = \int_0^1 \binom{x}{n}\,dx$. Thus we obtain

$$\int_0^1 f(x)\,dx = f(0) + \Delta f(0)\frac{b_1}{1!} + \Delta^2 f(0)\frac{b_2}{2!} + \cdots + \Delta^n f(0)\frac{b_n}{n!} + \cdots \tag{16}$$

(see [7, §96, Eq. (1)]). By Corollary 2.3, $b_n/n!$ equals $(-1)^{n-1} p_n/(n+1)$, where p_n is the probability that the second player has a winning strategy starting from a random position of rank n. Thus, we obtain the following result.

Theorem 5.3. *Let* $f(x)$ *be a polynomial function, or a function for which the expansion* (16) *converges. Consider the original Bernoulli game, scored by the sequence*

$$1, \Delta f(0)/2, -\Delta^2 f(0)/3, \ldots, (-1)^{n-1}\Delta^n f(0)/(n+1), \ldots.$$

As $n \to \infty$, *the expected gain of player B converges to* $\int_0^1 f(x)\,dx$.

Example 5.4. As noted by Jordan [7, §96], we have

$$\ln(2)=\int_0^1 \frac{dx}{x+1}=\sum_{n=0}^{\infty}\frac{(-1)^n b_n}{(n+1)!}=1-\sum_{n=1}^{\infty}\frac{p_n}{(n+1)^2}.$$

Thus B's expected gain is $\ln(2)$ dollars if at the beginning A pays 1 dollar to B (for $n=0$) and then, for each $n>0$, B starts a round at a random position of rank n and pays $1/(n+1)^2$ dollars for losing the nth round.

The defining equation (3), together with Corollaries 2.3 and 4.3, may also be used to construct scoring sequences for which the expected gain of player B may be calculated.

Theorem 5.5. *Let t be a real number and x be a non-negative integer, such that the expansion* (3) *converges. For $x=0$ let the players A and B play the original Bernoulli game, scored by $1, t/2, -t/3, \dots, (-1)^{n-1}t^n/(n+1), \dots,$ for $x>0$, let them play the polynomial Bernoulli game of the second kind, indexed with x, scored by $\{(-t)^n(x+n-1)_n/n!\}_{n=0}^{\infty}$. Then, as $n\to\infty$, the expected gain of player B converges to $t(1+t)^{-x}/\ln(1+t)$.*

Proof. Let us denote by p_n the probability that the second player in round n wins. If $x=0$ then, by Corollary 2.3, we have $b_n/n! = (-1)^{n-1}p_n/(n+1)$ for $n>0$, and we set $p_0=1$. If $x>0$ then, by Corollary 4.3 we have

$$\frac{b_n(-x)}{n!}=\frac{(-1)^n p_n (x+n-1)_n}{n!}\quad \text{for } n\geqslant 0.$$

Thus, for $x=0$ we have

$$\frac{t}{\ln(1+t)}=\sum_{n=0}^{\infty}\frac{b_n}{n!}t^n=1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}p_n}{n+1}t^n$$

and, for $x>0$, we have

$$\frac{t(1+t)^{-x}}{\ln(1+t)}=\sum_{n=0}^{\infty}\frac{b_n(-x)}{n!}t^n=1+\sum_{n=1}^{\infty}\frac{(-1)^n p_n(x+n-1)_n}{n!}t^n. \qquad \square$$

Example 5.6. Substituting $t=-1/2$ and $x=0$ gives that in the original Bernoulli game, scored by $1, -1/(2\cdot 2), -1/(4\cdot 3), \dots, -1/(2^n\cdot(n+1)), \dots,$ the expected gain of player B converges to $1/2\ln(2)$.

Remark 5.7. The scoring systems for which there is a d such that $c_n<0$ for all $n\geqslant d$, such as the ones in Examples 5.4 and 5.6, may be easily implemented as a casino game. At the beginning player A, the gambler, should pay $-\sum_{n=d}^{\infty} c_n\cdot p_n$ to the casino, that is, player B. After this, they start playing the appropriate Bernoulli game, scored by $\{c_n\}_{n=0}^{\infty}$, starting from round $n=d$. Since $c_n<0$, each round is started by B, and A can only win money in each round. If the game is played "to the end," B's expected gain is

$$-\sum_{n=d}^{\infty} c_n\cdot p_n+\sum_{n=d}^{\infty} c_n\cdot p_n=0$$

dollars. In practice, however, A will get tired before an infinite amount of time, so B can expect to have a positive gain. Of course, such a game is only interesting for A, if it is possible for A to win more money than the amount paid at the beginning. This is always true, since A has a chance to win $\sum_{n=d}^{\infty}(-c_n)$ dollars which is strictly more than the $-\sum_{n=d}^{\infty} c_n \cdot p_n$ that A has paid for the right to play the game. In our examples, however, it is also true that the gambler has a chance to win more than his or her payment right at the beginning. It should be noted, that $p_1 = 1$ in the original Bernoulli game, so B is bound to pay money to A in round 1. Thus, for the original Bernoulli game, we may always simplify the game by starting it at a round $n \geqslant 2$ and adjusting the gambler's payments accordingly. Subject to this adjustment, in Example 5.4, the gambler is supposed to pay $1 - 1/4 - \ln(2) \approx 0.06$ dollars, and in round $n = 2$ player A has a chance to win $1/9 \approx 0.11$ dollars, with a probability of $1/4 = 25\%$. In Example 5.6, the gambler should pay $1 - 1/4 - 1/(2\ln(2)) \approx 0.03$ dollars, and in round $n = 2$ the gambler has a chance to win $1/(4 \cdot 3) \approx 0.08$ dollars, with a probability of 25%. In all these games, the gambler may keep playing as long as he or she pleases.

Finally, we wish to mention that any polynomial $f(x)$ may be written as a combination of Bernoulli polynomials of the second kind. In fact, we have

$$f(x) = \frac{d}{dx}\Delta^{-1} f(0) \cdot b_0(x) + \sum_{n=1}^{\infty} \frac{d}{dx}\Delta^{n-1} f(0) \frac{b_n(x)}{n!}. \tag{17}$$

See Jordan [7, §97]. Eq. (17) contains only finitely many non-zero terms for a polynomial $f(x)$, but it remains valid for other continuous functions when the expression on the right-hand side converges. By considering the expansion of $\ln(x + 1)$ into a series of Bernoulli polynomials of the second kind, Jordan [7, §97, (11)] obtains the formula

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n \cdot n!} = C,$$

where C is Euler's constant. As a consequence, we obtain the following result.

Corollary 5.8. *In the original Bernoulli game, scored by*

$$0, \frac{1}{1 \cdot 2}, -\frac{1}{2 \cdot 3}, \ldots, (-1)^{n-1} \cdot \frac{1}{n \cdot (n+1)}, \ldots$$

the expected gain of player B converges to Euler's constant C.

Remark 5.9. Since $1/(n \cdot (n + 1)) = 1/n - 1/(n + 1)$, Corollary 5.8 has the following interesting reformulation. Let A and B play two rounds of the original Bernoulli game for each $n > 0$, starting with a random position of rank n. For odd n, player B starts the first round, for even rank, player A starts the first round of rank n. If the player starting the round wins, no money is paid, otherwise the player starting a round pays $1/n$ dollars for losing a first round of rank n and $1/(n + 1)$ dollars for losing a second round of rank n. This definition is "almost symmetric" in A and B, and B's very slight advantage is translated into an expected gain of C.

6. The degenerate Bernoulli game

In this section we construct for each rational $\lambda = p/q > 1$ a variant of the Bernoulli game that is related to the degenerate Bernoulli numbers $\beta_n(p/q, 0)$ in the same way as the Bernoulli numbers of the second kind are related to the original Bernoulli game. To achieve this goal, observe first that for $x = 0$, (6) may be rewritten as

$$\sum_{n=0}^{\infty} \beta_n(\lambda, 0)\frac{t^n}{n!} \cdot \sum_{n=1}^{\infty} \binom{1/\lambda}{n} \lambda^n t^n = t.$$

Comparing the coefficients of t yields $\beta_0(\lambda,0)=1$, comparing the coefficients of t^{n+1} for $n \geqslant 0$ yields

$$\frac{\beta_n(\lambda,0)}{n!}+\sum_{m=0}^{n-1}\frac{\beta_m(\lambda,0)}{m!}\binom{1/\lambda}{n+1-m}\lambda^{n+1-m}=0.$$

Using $\beta_0(\lambda,0)=1$ we may rearrange this equation as

$$\frac{\beta_n(\lambda,0)}{n!}+\sum_{m=1}^{n-1}\frac{\beta_m(\lambda,0)}{m!}\binom{1/\lambda}{n+1-m}\lambda^{n+1-m}=-\binom{1/\lambda}{n+1}\lambda^{n+1}. \tag{18}$$

Here, by $\lambda=p/q$, we have

$$\begin{aligned}\binom{1/\lambda}{k}\lambda^k &= \frac{1/\lambda(1/\lambda-1)\cdots(1/\lambda-k+1)}{k!}\lambda^k=\frac{(1-\lambda)\cdots(1-(k-1)\lambda)}{k!}\\ &= \frac{(1-p/q)\cdots(1-(k-1)p/q)}{k!}=\frac{(-1)^{k-1}(p-q)(2p-q)\cdots((k-1)p-q)}{k!q^{k-1}}\end{aligned}$$

for $k>0$. Using this observation, we may rewrite (18) as

$$\frac{\beta_n(\lambda,0)}{n!}+\sum_{m=1}^{n-1}\frac{\beta_m(\lambda,0)}{m!}\frac{(-1)^{n-m}(p-q)\cdots((n-m)p-q)}{(n+1-m)!q^{n-m}}=\frac{(-1)^n(p-q)\cdots(np-q)}{(n+1)!q^n}. \tag{19}$$

Introducing $\kappa_n:=(-q)^n(n+1)!\beta_n(\lambda,0)$ for $n\geqslant 1$ and multiplying both sides of (19) by $(-q)^n n!(n+1)!$ yields

$$\kappa_n+\sum_{m=1}^{n-1}\kappa_m\frac{n!}{(m+1)!}\binom{n+1}{n+1-m}(p-q)\cdots\big((n-m)p-q\big)=n!(p-q)\cdots(np-q). \tag{20}$$

Definition 6.1. Let $p>q>0$ be positive integers. The positions in *degenerate Bernoulli game indexed with* (p,q) are all pairs of words $(u_1\cdots u_n, v_1\cdots v_n)$ (where n is a positive integer) such that

(i) the letters $u_1,\dots,u_n$ and $v_1,\dots,v_n$ are positive integers;
(ii) for each $i\geqslant 1$ we have $1\leqslant u_i\leqslant i$ and $q+1\leqslant v_i\leqslant ip$.

A *valid move* consists of replacing the pair $(u_1\cdots u_n, v_1\cdots v_n)$ with $(u_1\cdots u_m, v_1\cdots v_m)$ for some $m\geqslant 1$ satisfying the following conditions:

(a) $u_{m+1}\leqslant\lceil\frac{v_j}{p}\rceil$ holds for $j=m+1,m+2,\dots,n$;
(b) If $\lceil\frac{v_{m+k}}{p}\rceil\leqslant\lceil\frac{v_{m+j}}{p}\rceil$ holds for $j=1,2,\dots,k$ then $\frac{v_{m+k}}{p}-\lfloor\frac{v_{m+k}}{p}\rfloor>\frac{q}{p}=\frac{1}{\lambda}=\mu$.

Here $\lceil x\rceil$ is the *ceiling* of x, i.e., the least integer that is not less than x and $\lfloor x\rfloor$ is the *floor* of x, i.e., the largest integer that is not more than x.

In analogy to the original Bernoulli game, one may define a partial order on the set of positions by taking initial segments of the words involved, and a rank function by taking the common length of the words in the pair. It is easy to verify that the resulting partially ordered set P and the function M induced by the definition of a valid move above satisfies the criteria given in Definition 3.1. It is worth noting that the inequalities on v_i's stated in (ii) are equivalent to requiring $v_i\geqslant q+1$ and $1\leqslant\lceil v_i/p\rceil\leqslant i$, thus the ordered pair $(u_1\cdots u_n, \lceil v_1/p\rceil\cdots\lceil v_n/p\rceil)$ must be a valid position in the original Bernoulli game. Condition (a) reiterates the condition of a valid move in the original Bernoulli game for the "factor positions" $\{(u_1 \cdots u_n, \lceil v_1/p \rceil \cdots \lceil v_n/p \rceil)\colon n \geqslant 1\}$, whereas condition (b) extends the "exception rule" set by $v_i \geqslant q+1$, by requiring the exclusion of the q smallest values of v_{m+k} allowed by the selection of $\lceil v_{m+k}/p \rceil$ each time when $\lceil v_{m+k}/p \rceil$ is a minimum in the subword $\lceil v_{m+1}/p \rceil \cdots \lceil v_{m+k}/p \rceil$.

Our main result is the following:

Theorem 6.2. *The number of kernel positions of rank n in the degenerate Bernoulli game indexed with (p, q) is $\kappa_n := (-q)^n (n+1)! \beta_n(p/q, 0)$.*

Proof. We may prove the theorem by showing that the numbers κ_n satisfy the recursion formula (20) for all $n > 0$. To prove (20), observe first that the right-hand side is the number of all positions of rank n in the game. There are exactly κ_n kernel positions of rank n, and for any other position there is a unique $m < n$ and a unique kernel position of rank m that is reachable from it. It is sufficient to show that any position of rank $m < n$ is reachable from exactly

$$\gamma_{m,n} = \frac{n!}{(m+1)!} \binom{n+1}{n+1-m} (p-q) \cdots \bigl((n-m)p - q\bigr)$$

positions of rank n. Given $u_1, \dots, u_m$ and $v_1, \dots, v_m$, we may select $u_{m+2}, \dots, u_n$ in exactly $n!/(m+1)!$ ways and these selections are independent of all other choices, since the definition of a valid move sets no condition these numbers. We are left to show that we may select the numbers $u_{m+1}, v_{m+1}, \dots, v_n$ in exactly $\binom{n+1}{n+1-m}(p-q) \cdots ((n-m)p-q)$ ways such that they satisfy the conditions in the definition of a valid move. As observed before stating this theorem, condition (a) reiterates the condition of a valid move of the original Bernoulli game for the numbers $u_{m+1}, \lceil v_{m+1}/p \rceil, \dots, \lceil v_n/p \rceil$. Using the bijection in the proof of Lemma 2.6 we may make each sequence $u_{m+1}, \lceil v_{m+1}/p \rceil + 1, \dots, \lceil v_n/p \rceil + 1$ correspond to an $(n-m+1)$-permutation of the set $\{1, \dots, n+1\}$. In particular, we may assign to each sequence $u_{m+1}, \lceil v_{m+1}/p \rceil + 1, \dots, \lceil v_n/p \rceil + 1$ an $(n-m+1)$-element subset $\{x_1, \dots, x_{n-m+1}\}$ of $\{1, \dots, n+1\}$ in such a way that the selection of the least element x_{n-m+1} is determined by the value of u_{m+1}. This is not a bijection any more, but we may use this map to define the selection of a sequence $u_{m+1}, v_{m+1}, \dots, v_n$ satisfying both conditions (a) and (b) as a two-phase process as follows:

(1) Select the $(n-m+1)$ element subset of $\{1, \dots, n+1\}$ that is permuted by the image of $u_{m+1}, \lceil v_{m+1}/p \rceil + 1, \dots, \lceil v_n/p \rceil + 1$ under the operation defined in the proof of Lemma 2.6;
(2) Choose the values of $v_n, v_{n-1}, \dots, v_{m+1}$ in this order in such a way that the selection is compatible with the selection made in the first step. (Note that u_{m+1} is already defined by the selection made in Phase 1.)

In Phase 1 there are exactly $\binom{n+1}{n-m+1}$ options. Thus it is sufficient to show that in Phase 2, the value of v_{m+k} can be fixed in exactly $k \cdot p - q$ ways for $k = 1, 2, \dots, n-m$, no matter how we chose v_{m+j} for $j > k$. At the beginning of Phase 2, we are given a subset $X := \{x_1, x_2, \dots, x_{n-m+1}\}$ of $\{1, \dots, n+1\}$ in such a way that x_{n-m+1} is the least element in X. The choice of $\lceil v_n/p \rceil$ determines the choice of x_1, which may be any of the $(n-m)$-elements of $X \setminus \{x_{n-m+1}\}$. Disregarding condition (b), this allows $(n-m)p$ possible values for v_n in most cases, $(n-m)p - q$ values if $v_n = 1$ is an allowed selection, but then condition (b) imposes no further restrictions. The element x_1 is the least element of $X \setminus \{x_{n-m+1}\}$ exactly when $\lceil v_n/p \rceil \leqslant \lceil v_{m+j} \rceil$ holds for $j = 1, 2, \dots, n-m$. Thus we may rephrase condition (b) as follows: if x_1 is the least letter in the subword $x_1 \cdots x_{n-m}$ of the $(n-m+1)$-permutation associated to $u_{m+1}, \lceil v_{m+1}/p \rceil + 1, \dots, \lceil v_n/p \rceil + 1$ then we must have $v_n/p - \lfloor v_n/p \rfloor > q/p$. This excludes q possible values from the $(n-m)p$ values of v_n allowed before. Therefore there are $(n-m)p - q$ ways to fix the value of v_n. Assume now that we have chosen the value of v_{m+j} for $j > k$, and thus selected $x_1, \dots, x_{n-m-k} \in X$. The choice of v_{m+k} encodes the choice of $x_{n+1-m-k}$ which may be any of the k elements in $X \setminus \{x_{n-m+1}, x_1, x_2, \dots, x_{n-m-k}\}$. Disregarding condition (b), this allows kp possible values for v_{m+k} in most cases, $kp-q$ values if $v_k=1$ is an allowed selection, but then condition (b) imposes no further restrictions. The element $x_{n+1-m-k}$ is the least element of $X\setminus\{x_{n-m+1},x_1,x_2,\dots,x_{n-m-k}\}$ exactly when $\lceil v_{m+k}/p\rceil \leqslant \lceil v_{m+j}\rceil$ holds for $j=1,2,\dots,k$. Thus we may rephrase condition (b) as follows: if $x_{n+1-m-k}$ is the least letter in the subword $x_{n+1-m-k}\cdots x_{n-m}$ of the $(n-m+1)$-permutation associated to $u_{m+1}, \lceil v_{m+1}/p\rceil+1,\dots,\lceil v_n/p\rceil+1$ then we must have $v_{m+k}/p-\lfloor v_{m+k}/p\rfloor<q/p$. This excludes q possible values from the $(n-m)p$ values of v_n allowed before. Therefore there are $kp-q$ ways to fix the value of v_{m+k}. $\square$

Corollary 6.3. *Assume that the starting position of the degenerate Bernoulli game indexed with (p,q) is selected at random among all positions of rank n (n is fixed), according to the uniform distribution. Then the probability that the game starts with a kernel position is*

$$p_n=\frac{(-q)^n(n+1)!\beta_n(p/q,0)}{n!(p-q)\cdots(np-q)}=\frac{(n+1)\beta_n(p/q,0)}{(1-p/q)(1-2p/q)\cdots(1-np/q)}.$$

Remark 6.4. Theorem 6.2 implies that $(-q)^n(n+1)!\beta_n(p/q,0)$ is a non-negative integer for $n\geqslant 1$. For $q=1$, i.e. positive integer λ, the same sign rule is stated for $\beta_n(p,0)$ by Young [18], right after Eq. (3.15). Young also proves that $p\beta_n(p,0)$ is an integer for an integer p.

7. Simple Bernoulli games

In this section we introduce two games that are simpler than the original Bernoulli game and calculate the number of their kernel positions of rank n. The first game is a "one-dimensional" variant of the original Bernoulli game.

Definition 7.1. The *flat Bernoulli game* is played on the partially ordered set of all words $u_1\cdots u_n$ of positive integers such that $1\leqslant u_i\leqslant i$ holds for $1\leqslant i$. The partial order is defined by $u_1\cdots u_m<u_1\cdots u_n$ for all $m<n$. A word $\{u_1,\dots,u_m\}$ belongs to $M(u_1\cdots u_n)$ exactly when

$$u_{m+1}<\min\{u_{m+2},\dots,u_n\}. \tag{21}$$

Condition (i) of Definition 3.1 is obviously satisfied, the proof of condition (ii) is completely analogous to Lemma 2.5. In analogy to Lemma 2.7 we have the following.

Lemma 7.2. *Any position $u_1\cdots u_m$ of rank m may be reached from exactly $(n)_{n-m}/(n-m)$ positions of rank n in a single move.*

In fact, by Lemma 2.6, the set of words $u_{m+1}\cdots u_n$ satisfying (21) is in bijection with those $(n-m)$-permutations of $1,\dots,n$ for which the last number is the least. Since $|P_n|=n!$, Eq. (10) specializes to

$$n!=\kappa_n+\sum_{m=1}^{n-1}\kappa_m\cdot\frac{(n)_{n-m}}{n-m}.$$

Dividing both sides by $n!$ yields

$$1=\frac{\kappa_n}{n!}+\sum_{m=1}^{n-1}\frac{\kappa_m}{m!}\cdot\frac{1}{n-m}. \tag{22}$$

The first 9 values of $\kappa_n/n!$ are listed in Table 1.

Table 1
The numbers $\kappa_n/n!$ for the flat Bernoulli game.

n	1	2	3	4	5	6	7	8	9
$\kappa_n/n!$	1	0	1/2	1/6	1/3	13/60	97/360	570/2520	1217/5040

Introducing

$$f(t) := \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} t^n,$$

multiplying both sides by t^n in (22) and summing over n yields

$$\sum_{n=1}^{\infty} t^n = f(t) + f(t) \cdot \sum_{n=1}^{\infty} \frac{t^n}{n}, \quad \text{that is,}$$

$$\frac{t}{1-t} = f(t) \cdot \bigl(1 - \ln(1-t)\bigr).$$

Thus we have shown

Proposition 7.3. *Assume that the starting position of the flat Bernoulli game is selected at random among all positions of rank n (n is fixed), according to the uniform distribution. Then the probability that the game starts with a kernel position (and thus the second player has a winning strategy) is the coefficient of t^n in*

$$f(t) := \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} t^n = \frac{t}{(1-t)(1-\ln(1-t))}.$$

Using the decomposition $f(t) = t/(1-t) \cdot 1/(1-\ln(1-t))$ we may write

$$\frac{\kappa_n}{n!} = \sum_{m=1}^{n} (-1)^m \cdot \frac{a_m}{m!},$$

where $a_m/m!$ is the coefficient of t^m in $1/(1-\ln(1+t))$. These numbers a_m are listed as sequence A00652 in the On-Line Encyclopedia of Integer Sequences [13], where they are indicated to be related to several combinatorially interesting sequences.

The second game is an “x-analogue” of the flat Bernoulli game, which turns out to be associated to a fairly sophisticated function, considering the simplicity of the rules.

Definition 7.4. Let $x > 0$ be a positive integer. The *simple x-Bernoulli game* is played on the partially ordered set of all words $u_1 \cdots u_n$ of positive integers such that $1 \leqslant u_i \leqslant x$ holds for $1 \leqslant i$. The partial order is defined by $u_1 \cdots u_m < u_1 \cdots u_n$ for all $m < n$. A word $\{u_1, \ldots, u_m\}$ belongs to $M(u_1 \cdots u_n)$ exactly when

$$u_{m+1} < \min\{u_{m+2}, \ldots, u_n\}. \tag{23}$$

It is easy to show the following.

Proposition 7.5. *Let $x > 0$ be a positive integer. The number of kernel positions κ_n of rank n in the simple x-Bernoulli game satisfies the recursion formula*

$$x^n = \kappa_n + \kappa_{n-1} \cdot x + \sum_{m=0}^{n-2} \kappa_m \sum_{u=1}^{x-1} (x-u)^{n-m-1} \quad \textit{for } n \geqslant 0.$$

In fact, x^n is the total number of positions of rank n. The summation over u corresponds to selecting $u_{m+1} = u$, the power $(x-u)^{n-m-1}$ corresponds, to selecting $u_{m+2}, \dots, u_n$. The details are left to the reader. Multiplying both sides of the recursion formula in Proposition 7.5 by t^n, and summing over all non-negative values of n yields

$$\frac{1}{1-xt} = \sum_{n=0}^{\infty} \kappa_n t^n \left(1 + xt + \sum_{n=2}^{\infty} \sum_{j=1}^{x-1} j^{n-1} t^n\right).$$

Here

$$1 + xt + \sum_{n=2}^{\infty} \sum_{j=1}^{x-1} j^{n-1} t^n = 1 + t + \sum_{j=1}^{x-1} \sum_{n=1}^{\infty} j^{n-1} t^n = 1 + t + \sum_{j=1}^{x-1} \frac{t}{1-jt}.$$

Thus we obtain

$$\sum_{n=0}^{\infty} \kappa_n t^n = \frac{1}{1-xt} \cdot \frac{1}{1 + t - \sum_{j=1}^{x-1} (j - 1/t)^{-1}}.$$

The sum $\sum_{j=1}^{x-1} (j - 1/t)^{-1}$ has a compact expression in terms of a shifted logarithmic derivative of the *gamma function*

$$\Gamma(z) := \int_0^{\infty} e^{-t} t^{z-1}\, dt,$$

called the *digamma function* $\digamma(x)$, given by

$$\digamma(x) := \frac{d}{dx} \ln\bigl(\Gamma(x+1)\bigr).$$

In fact, the digamma function has the expansion

$$\digamma(x) = -C + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{x+n}\right),$$

where C is Euler's constant (see [7, §19, (7)]), from which it is easy to deduce

$$\sum_{j=1}^{x-1} \frac{1}{j - 1/t} = \digamma\left(x - 1 - \frac{1}{t}\right) - \digamma\left(-\frac{1}{t}\right).$$

Using this equation we may rephrase Proposition 7.5 as follows.

Theorem 7.6. *Let $x > 0$ be a positive integer. Then the numbers of kernel positions κ_n of rank n in the simple x-Bernoulli game satisfy the equation*

$$\sum_{n=0}^{\infty} \kappa_n t^n = \frac{1}{1-xt} \cdot \frac{1}{1+t-F(x-1-\frac{1}{t})+F(-\frac{1}{t})}.$$

8. Concluding remarks

All Bernoulli games in this paper have the common property that they are played on (pairs, triplets of) words, and each valid move involves taking initial segments. This is far from exhausting all possibilities satisfying the definition of a pair (P, M) introducing a Bernoulli type game. Just to mention one example, consider the set of all words $u_1 \cdots u_n$, satisfying $1 \leqslant u_i \leqslant x$ for all i and for some fixed x, and the *subword order* in which a word $\underline{u} := u_1 \cdots u_m$ is less than $\underline{v} := v_1 \cdots v_n$ if $\underline{u}$ is obtained from $\underline{v}$ by removing some letters and keeping the order of the remaining letters intact. Let us define a valid move as a removal of all letters greater than equal to y from a word $\underline{v}$ if all letters that are greater than y in $\underline{v}$ are to the right of the rightmost y. It is easy to verify that this definition yields a Bernoulli type game. It is yet to be seen whether other combinatorial objects allow the definition of a pair (P, M) introducing a Bernoulli type game, and which of these yield interesting generating functions for the numbers of kernel positions. Furthermore, it should be noted that the truly interesting requirements in Definition 3.1 are conditions (i) and (ii): these by themselves guarantee that from each non-kernel position a unique kernel position is reachable and that the winning strategy from a given starting position may be described by the results in Section 3. It is conceivable that, in some situations, condition (iii) will be omitted or replaced by a weaker condition to provide a combinatorial model for a more complicated generating function.

The scoring systems introduced in Section 5 are worth a second look using deeper results of analysis and probability theory. For the moment, we just stated the expected gains, without any consideration to the speed of convergence of the underlying series or the properties of the underlying random variables.

It is desirable to have a simpler model for the Bernoulli polynomials of the first kind in our setting, and to find any model for non-zero integer substitutions into the degenerate Bernoulli polynomials. Finding the best models for all these might lead to a deeper understanding of the interactions between the Bernoulli polynomials and their degenerate generalizations.

Even the original Bernoulli game deserves a second look. The number of words $z_1 \cdots z_n$ satisfying $1 \leqslant z_i \leqslant i$ is $n!$, and the proof of Lemma 2.6 indicates how to establish a bijection between all such words and all permutations $x_1 \cdots x_n$ of the set $\{1, \ldots, n\}$: x_1 is $z_n \in \{1, \ldots, n\}$ and, for all $j > 1$, x_j is the z_{n+1-j}th largest element in $\{1, \ldots, n\} \setminus \{x_1, \ldots, x_{j-1}\}$. Thus the original Bernoulli game may be considered as a game on pairs of permutations. Jordan [7, §89, (7)] has the following formula connecting his Bernoulli numbers of the second kind with the Stirling numbers of the first kind:

$$\psi_n(0) = \frac{1}{n!} \sum_{m=1}^{n} \frac{s(n, m)}{m+1} \tag{24}$$

(summing to $n+1$ is unnecessary). The number $(-1)^{n-m} s(n, m)$ equals the number of permutations of n elements with m cycles or m left-to-right minima (see Stanley [14, §1.3]). Perhaps a deeper analysis of the original Bernoulli game could help explain this formula combinatorially. We may also be interested in finding the winning positions of rank n. Starting the game in such a position $(u_1 \cdots u_n, v_1 \cdots v_n)$ the first player loses instantly, since there is no valid move. For such positions the word $v_1 \cdots v_n$ must satisfy

$$\min\{v_{m+1}, \ldots, v_n\} \leqslant m \quad \text{for all } m \geqslant 1, \tag{25}$$

otherwise $u_{m+1} \leqslant \min\{v_{m+1}, \ldots, v_n\}$ automatically follows from $u_{m+1} \leqslant m+1$. Conversely, if $(v_1 \cdots v_n)$ satisfies (25) then $(1 \cdots n, v_1 \cdots v_n)$ is a winning position. Using the bijection indicated in Lemma 2.6, the words $v_1 \cdots v_n$ satisfying (25) are in bijection with those permutations of $\{1, \ldots, n\}$ which take no proper interval of the form $[m+1, n]$ into itself. Replacing each x_i with $n+1-x_i$ we find ourselves interested in the class of permutations of $\{1, 2, \ldots, n\}$ which take no proper interval of the form $[1, i]$ into itself. Such permutations form a basis for the Malvenuto–Reutenauer Hopf-algebra, introduced by Malvenuto and Reutenauer in [9], as this was shown by Poirier and Reutenauer [11]. For further detailed study of the Malvenuto–Reutenauer Hopf algebra we refer the reader to the work of Aguiar and Sottile [2]. The question naturally arises, whether there is an algebra or coalgebra properly containing the Malvenuto–Reutenauer Hopf algebra whose basis consists of the kernel positions (or at least the winning positions) of rank n of the original Bernoulli game.

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